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The uniqueness of an orthogonality measure

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Abstract

Given two finite sets of real points $\{z_{n,j}\}_{j=1}^{n}$ and $\{z_{n+1,j}\}_{j=1}^{n+1}$ satisfying the interlacing property $z_{n+1,j} < z_{n,j} < z_{n+1,j+1}$, $j = 1, \ldots, n$, the monic polynomials $p_n(z) = \prod_{j=1}^{n}(z - z_{n,j})$ and $p_{n+1}(z) = \prod_{j=1}^{n+1}(z - z_{n+1,j})$ can be embedded in an infinite sequence of monic orthogonal polynomials (cf. [B. Wendroff, On orthogonal polynomials, Proc. Amer. Math. Soc. 12 (1961) 554–555. [6]). We discuss, in turn, the uniqueness of the measure of orthogonality μ arising in this context if card $(\mathrm{supp}(\mu)) = n + 1$ or $n + 2$.

MSC: 42C05

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1. Introduction

In [6], Wendroff showed that if two finite sets of real points $\{z_{n,1}, \ldots, z_{n,n}\}$ and $\{z_{n+1,1}, \ldots, z_{n+1,n+1}\}$ satisfy the interlacing property

$$z_{n+1,j} < z_{n,j} < z_{n+1,j+1}, \qquad j = 1, \ldots, n, \tag{1.1}$$

then the polynomials $p_n(z) = \prod_{j=1}^{n}(z - z_{n,j})$ and $p_{n+1}(z) = \prod_{j=1}^{n+1}(z - z_{n+1,j})$ can be embedded in a sequence of orthogonal polynomials $\{p_n\}_{n=0}^{\infty}$. His construction in [6] first defines the polynomials p_{n-j} for $j = 1, 2, \ldots, n - 1$ in such a way that ensures that $p_0, p_1, \ldots, p_{n+1}$ is a finite orthogonal sequence, taking into account that $p_0 \equiv 1$. He then extends this to an infinite orthogonal sequence $\{p_n\}_{n=0}^{\infty}$ by defining p_k for $k > n + 1$ via a three-term recurrence relation of the form

$$p_k(z) = (z + a_k)p_{k-1}(z) - b_k p_{k-2}(z), \qquad k = n + 2, n + 3, \ldots, \tag{1.2}$$

where $a_k, b_k \in \mathbb{R}$ and $b_k > 0$ for $k = n + 2, n + 3, \ldots$. Having defined $\{p_n\}_{n=0}^{\infty}$ in this way, Favard's theorem [2] then guarantees the existence of a probability measure μ with respect to which $\{p_n\}_{n=0}^{\infty}$ is orthogonal. Clearly, μ will depend on the choice of the coefficients a_k and b_k for $k > n + 1$ in (1.2) and one cannot expect uniqueness of the measure of orthogonality μ.

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In this paper, we prove that if $z_{n+1,j} < z_{n,j} < z_{n+1,j+1}$, $j = 1, \ldots, n$ then there exists a uniquely determined probability measure μ_0, with card $(\mathrm{supp}(\mu_0)) = n+1$, with respect to which $p_n(z) = \prod_{j=1}^{n}(z - z_{n,j})$ and $p_{n+1}(z) = \prod_{j=1}^{n+1}(z - z_{n+1,j})$ are orthogonal. Our proof does not invoke Favard's theorem and Wendroff's theorem follows as an immediate corollary. We also consider sufficient conditions that ensure the uniqueness of the orthogonality measure μ arising in this context when card $(\mathrm{supp}(\mu)) = n + 2$.

2. Results

We shall use the following notation. Let

$$p_n(z) = z^n + p_{n,1}z^{n-1} + \cdots + p_{n,n} = \prod_{j=1}^{n}(z - z_{n,j}) \tag{2.1}$$

be the real monic polynomial with distinct real zeros $z_{n,j}$, $j = 1, \ldots, n$. Let δ_x denote the counting measure with mass 1 at the point x.

Theorem 2.1. *Let*

$$z_{n+1,j} < z_{n,j} < z_{n+1,j+1}, \quad j = 1, \ldots, n. \tag{2.2}$$

Then there exists a unique probability measure $\mu_0 = \sum_{j=1}^{n+1} w_j \delta_{z_{n+1,j}}$ *with* $w_j > 0$, $j = 1, \ldots, n + 1$ *such that*

$$\int x^{\ell} p_n(x)\, \mathrm{d}\mu_0(x) = 0, \quad \ell = 0, 1, \ldots, n - 1, \tag{2.3}$$

where $p_n(z)$ *is given by* (2.1).

Corollary 2.2. *Let* (2.2) *hold. Then the polynomials*

$$p_n(z) = \prod_{j=1}^{n}(z - z_{n,j}) \quad \text{and} \quad p_{n+1}(z) = \prod_{j=1}^{n+1}(z - z_{n+1,j})$$

can be embedded in an orthogonal sequence $\{p_n\}_{n=0}^{\infty}$.

Corollary 2.3. *Let* (2.2) *hold and let* p_n, p_{n+1} *be given by* (2.1). *If* μ *is any probability measure with* card $(\mathrm{supp}(\mu)) = n + 1$ *and* p_n, p_{n+1} *are orthogonal with respect to* μ, *then* $\mu = \mu_0$.

Theorem 2.4. *Let* (2.2) *hold and let* p_n, p_{n+1} *be given by* (2.1). *If* μ *is a probability measure supported on* $\mathbb{R}$ *with* card $(\mathrm{supp}(\mu)) = n + 2$ *and* p_n, p_{n+1} *are orthogonal with respect to* μ, *then* μ *is uniquely determined if two different points of* $\mathrm{supp}(\mu)$ *are known.*

Corollary 2.5. *Let* (2.2) *hold and let* $x_1, x_2 \in \mathbb{R}$ *with* $x_1 < z_{n+1,1} < \cdots < z_{n+1,n+1} < x_2$. *Then there exists a unique probability measure* μ, *with* p_n, p_{n+1} *orthogonal with respect to* μ, *which is supported on* $\mathbb{R}$ *with* card $(\mathrm{supp}(\mu)) = n+2$ *and* $x_1, x_2 \in \mathrm{supp}(\mu)$.

3. Related results

Theorem 2.1 can be deduced from more general results obtained in [1] by de Boor and Saff who answer the following question: given any two finite sets of real points $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_{k-1}\}$, if $P(z) = \prod_{j=1}^{n}(z - x_j)$ and $Q(z) = \prod_{j=1}^{k-1}(z - y_j)$, when are the polynomials P and Q members of the same sequence of orthogonal polynomials? They also find all possible weights supported on n points that constitute the probability measure of orthogonality. In [5], Vinet and Zhedanov use the notion of de Boor–Saff duality, developed in [1], to give a characterisation of classical and semi-classical orthogonal polynomials from their dual polynomials. For a discussion on these and related ideas, see also [3]. In a different but related development, Marcellan and Alvarez-Nodarse [4] consider extensions of Favard's theorem to sequences of monic polynomials that satisfy a recurrence relation different from (1.2).

4. Proofs

Proof of Theorem 2.1. The conditions (2.3) are equivalent to the linear homogeneous system

$$\sum_{j=1}^{n+1} w_j z_{n+1,j}^{\ell} p_n(z_{n+1,j}) = 0, \quad \ell = 0, 1, \ldots, n-1, \tag{4.1}$$

with $w_1, \ldots, w_{n+1}$ as unknowns. Since there are n equations and $(n+1)$ unknowns, a non-trivial solution exists. Define n polynomials

$$q_{n,\ell}(z) = \frac{p_{n+1}(z)}{(z - z_{n+1,\ell})(z - z_{n+1,\ell+1})}, \quad \ell = 1, \ldots, n.$$

Then $q_{n,\ell}(z)$ is a polynomial of degree $(n-1)$ for each $\ell = 1, \ldots, n$ and

$$q_{n,\ell}(z_{n+1,j}) = \begin{cases} 0 & \text{for } j \neq \ell,\ j \neq \ell+1, \\ \dfrac{p'_{n+1}(z_{n+1,\ell})}{(z_{n+1,\ell} - z_{n+1,\ell+1})}, & j = \ell, \\ \dfrac{p'_{n+1}(z_{n+1,\ell+1})}{(z_{n+1,\ell+1} - z_{n+1,\ell})}, & j = \ell+1. \end{cases}$$

Since the linear span of $\{q_{n,1}, \ldots, q_{n,n}\}$ is equal to the linear span of $\{1, z, \ldots, z^{n-1}\}$, the system of equations (4.1) is equivalent to

$$\sum_{j=1}^{n+1} w_j q_{n,\ell}(z_{n+1,j}) p_n(z_{n+1,j}) = 0, \quad \ell = 1, \ldots, n. \tag{4.2}$$

If we denote the coefficient matrix of the system of equations (4.2) by

$$A = (a_{\ell,j})_{\ell=1;\,j=1}^{n;\,n+1},$$

we see that A is an upper bidiagonal matrix with non-zero entries

$$a_{\ell,\ell} = \frac{p'_{n+1}(z_{n+1,\ell}) p_n(z_{n+1,\ell})}{(z_{n+1,\ell} - z_{n+1,\ell+1})}, \tag{4.3}$$

$$a_{\ell,\ell+1} = \frac{p'_{n+1}(z_{n+1,\ell+1}) p_n(z_{n+1,\ell+1})}{(z_{n+1,\ell+1} - z_{n+1,\ell})} \tag{4.4}$$

for $\ell = 1, \ldots, n$. Solving (4.2) for $w_1, \ldots, w_{n+1}$, we have $a_{1,1}w_1 + a_{1,2}w_2 = 0, \ldots,$ so that

$$w_2 = -\frac{a_{1,1}}{a_{1,2}} w_1, \quad w_3 = -\frac{a_{2,2}}{a_{2,3}} w_2, \ldots, w_{n+1} = -\frac{a_{n,n}}{a_{n,n+1}} w_n,$$

or

$$\begin{aligned} w_k &= (-1)^{k+1} \prod_{s=1}^{k-1} \frac{a_{s,s}}{a_{s,s+1}} w_1 \\ &= \frac{p'_{n+1}(z_{n+1,1}) p_n(z_{n+1,1})}{p'_{n+1}(z_{n+1,k}) p_n(z_{n+1,k})} w_1, \quad k = 2, \ldots, n+1. \end{aligned} \tag{4.5}$$

The interlacing property (2.2) guarantees that each of the polynomials p'_{n+1} and p_n have a different sign at successive zeros of p_{n+1} and it follows from (4.5) that $w_1, \ldots, w_{n+1}$ have the same sign. Finally since $1 = \|\mu_0\| = w_1 + \cdots + w_{n+1}$, it follows that $w_i > 0$, $i = 1, \ldots, n+1$ are uniquely determined. □

Remark. The weights w_k derived in (4.5) can be transformed into the formula (see [5], Eq. (1.11))

$$w_k = \frac{h_n}{p'_{n+1}(z_{n+1,k}) p_n(z_{n+1,k})},$$

where

$$\sum_{k=1}^{n+1} w_k p_i(z_{n+1,k}) p_j(z_{n+1,k}) = h_i \delta_{ij}, \quad i, j = 1, \ldots, n$$

and

$$\sum_{k=1}^{n+1} w_k = h_0 = 1.$$

Proof of Corollary 2.2. The construction of the measure μ_0 in Theorem 2.1 ensures that $p_{n+1} = \prod_{j=1}^{n+1}(z - z_{n+1,j})$ is orthogonal with respect to μ_0. Since μ_0 is uniquely determined and p_n, p_{n+1} are orthogonal with respect to μ_0, the proof is complete. □

Proof of Corollary 2.3. If card $(\mathrm{supp}(\mu)) = n+1$ and p_{n+1} is orthogonal with respect to μ, then the two sets $\mathrm{supp}(\mu)$ and $Z(p_{n+1})$ are the same.

Therefore $\mu = \sum_{j=1}^{n+1} \tilde{w}_j \delta_{z_{n+1,j}}$ for some $\tilde{w}_j$, $j = 1, \ldots, n+1$. Since μ is a probability measure and p_n is orthogonal with respect to μ, it follows from Theorem 2.1 that $\tilde{w}_j = w_j$ are uniquely determined for $j = 1, \ldots, n+1$ and therefore $\mu = \mu_0$. □

Proof of Theorem 2.4. Suppose μ is any measure satisfying the conditions of Theorem 2.4 with

$$\mathrm{supp}(\mu) = \{z_{n+2,1}, \ldots, z_{n+2,n+2}\} \subseteq \mathbb{R}.$$

Then $p_{n+2}(z) = \prod_{j=1}^{n+2}(z - z_{n+2,j})$ is orthogonal with respect to μ, as are p_n and p_{n+1} by assumption, and therefore these polynomials satisfy a three-term recurrence relation of the form

$$p_{n+2}(z) = (z + a_n) p_{n+1}(z) - b_n p_n(z), \quad a_n \in \mathbb{R}, \quad b_n > 0.$$

Moreover, the interlacing property, namely,

$$z_{n+2,j} < z_{n+1,j} < z_{n+2,j+1}, \quad j = 1, \ldots, n+1$$

holds where $\{z_{n,1}, \ldots, z_{n,n}\}$ are the zeros of $p_n(z)$ in increasing order.Now, suppose μ and $\widetilde{\mu}$ are two measures satisfying the conditions of Theorem 2.4 with

$$\mathrm{supp}(\mu) = \{z_{n+2,1}, \ldots, z_{n+2,n+2}\},$$

$$p_{n+2}(z) = \prod_{j=1}^{n+2}(z - z_{n+2,j}) = (z + a_n) p_{n+1}(z) - b_n p_n(z)$$

and

$$\mathrm{supp}(\widetilde{\mu}) = \{\widetilde{z}_{n+2,1}, \ldots, \widetilde{z}_{n+2,n+2}\},$$

$$\widetilde{p}_{n+2}(z) = \prod_{j=1}^{n+2}(z - \widetilde{z}_{n+2,j}) = (z + \widetilde{a}_n) p_{n+1}(z) - \widetilde{b}_n p_n(z)$$

and suppose that

$$x_i \in \operatorname{supp}(\mu) \cap \operatorname{supp}(\widetilde{\mu}), \quad i = 1, 2, \quad x_1 \neq x_2.$$

Then

$$0 = p_{n+2}(x_i) = (x_i + a_n) p_{n+1}(x_i) - b_n p_n(x_i), \quad i = 1, 2, \tag{4.6}$$

$$0 = \widetilde{p}_{n+2}(x_i) = (x_i + \widetilde{a}_n) p_{n+1}(x_i) - \widetilde{b}_n p_n(x_i), \quad i = 1, 2. \tag{4.7}$$

Subtracting we obtain the equations

$$(a_n - \widetilde{a}_n) p_{n+1}(x_1) - (b_n - \widetilde{b}_n) p_n(x_1) = 0,$$

$$(a_n - \widetilde{a}_n) p_{n+1}(x_2) - (b_n - \widetilde{b}_n) p_n(x_2) = 0,$$

which have the unique solution $a_n = \widetilde{a}_n$, $b_n = \widetilde{b}_n$, provided $p_{n+1}(x_1) p_n(x_2) - p_{n+1}(x_2) p_n(x_1) \neq 0$. We know that $p_{n+1}(x_i) \neq 0$ for $i = 1, 2$ and if $p_n(x_i) \neq 0$ for $i = 1, 2$, then

$$p_{n+1}(x_1) p_n(x_2) - p_{n+1}(x_2) p_n(x_1) = 0$$

implies $p_{n+1}(x_1) = p_{n+1}(x_2) p_n(x_1)/p_n(x_2)$. From this and (4.6) with $i = 1$, we see that

$$0 = (x_1 + a_n) \frac{p_{n+1}(x_2) p_n(x_1)}{p_n(x_2)} - b_n p_n(x_1),$$

or

$$0 = (x_1 + a_n) p_{n+1}(x_2) - b_n p_n(x_2).$$

Also, from (4.6) with $i = 2$, we have

$$0 = (x_2 + a_n) p_{n+1}(x_2) - b_n p_n(x_2),$$

and subtracting the last two equations leads to $(x_1 - x_2) p_{n+1}(x_2) = 0$ which implies $x_1 = x_2$, a contradiction. It remains to show that if $p_n(x_i) = 0$ for $i = 1$ and/or $i = 2$, we still have a unique solution. First, if $p_n(x_1) = 0$ but $p_n(x_2) \neq 0$ (or vice versa), then $p_{n+1}(x_1) p_n(x_2) - p_{n+1}(x_2) p_n(x_1) \neq 0$ and we are done. If $p_n(x_1) = p_n(x_2) = 0$, then (4.6) with $i = 1, 2$ in turn yields $x_1 = -a_n = x_2$, a contradiction.

We have now established that, under the conditions of Theorem 2.4, $p_{n+2}(z)$ (and therefore its zeros $\{z_{n+2,1}, \ldots, z_{n+2,n+2}\}$), is uniquely determined by the three-term recurrence relation. The uniqueness of the measure μ now follows in the same way as in the proof of Theorem 2.1, where here $\mu_0 = \sum_{j=1}^{n+2} w_j \delta_{z_{n+2,j}}$, the polynomial $p_{n+1}(z)$ is orthogonal with respect to μ_0, the required interlacing property is satisfied and $w_j > 0$, $j = 1, \ldots, n+2$ are uniquely determined. This completes the proof. □

Proof of Corollary 2.5. Let $p_{n+2}(z)$ be defined by

$$p_{n+2}(z) = (z + a_n) p_{n+1}(z) - b_n p_n(z), \tag{4.8}$$

where

$$a_n = \frac{x_1 p_{n+1}(x_1) p_n(x_2) - x_2 p_n(x_1) p_{n+1}(x_2)}{p_n(x_1) p_{n+1}(x_2) - p_{n+1}(x_1) p_n(x_2)}$$

and

$$b_n = \frac{(x_1 - x_2) p_{n+1}(x_1) p_{n+1}(x_2)}{p_n(x_1) p_{n+1}(x_2) - p_{n+1}(x_1) p_n(x_2)}.$$

We claim that a_n and $b_n \in \mathbb{R}$ are well defined and $b_n > 0$. Indeed, since neither p_n nor p_{n+1} changes sign outside the interval $(z_{n+1,1}, z_{n+1,n+1})$, the behaviour as $x \to \pm\infty$ of these two polynomials for n even and n odd shows that

$$p_n(x_1) p_{n+1}(x_2) - p_{n+1}(x_1) p_n(x_2) \neq 0$$

and

$$p_{n+1}(x_1)p_{n+1}(x_2)/(p_n(x_1)p_{n+1}(x_2) - p_n(x_2)p_{n+1}(x_1)) < 0$$

for all $n \in \mathbb{N}$. Since $x_1 < x_2$, we see from (4.10) that $b_n > 0$ as we have claimed.

Substituting (4.9) and (4.10) into (4.8), we easily obtain $p_{n+2}(x_1) = 0 = p_{n+2}(x_2)$. Also, for each $j \in \{1, \ldots, n+1\}$, we have

$$p_{n+2}(z_{n+1,j}) = -b_n p_n(z_{n+1,j})$$

and since $p_n(z_{n+1,j})p_n(z_{n+1,j+1}) < 0$ for each $j = 1, \ldots, n$ and $b_n \neq 0$, it follows that $p_{n+2}(z)$ has at least one zero of odd order in each of the n intervals $(z_{n+1,j}, z_{n+1,j+1})$, $j = 1, \ldots, n$. Therefore $p_{n+2}(z)$ has $(n+2)$ real simple zeros, $\{z_{n+2,j}\}_{j=1}^{n+2}$, with $x_1 = z_{n+2,1}$ and $x_2 = z_{n+2,n+2}$ satisfying

$$z_{n+2,j} < z_{n+1,j} < z_{n+2,j+1}, \quad j = 1, \ldots, n+1.$$

Let $\mu = \sum_{j=1}^{n+2} w_j \delta_{z_{n+2,j}}$ be the probability measure with mass points at the zeros of p_{n+2}. Then card $(\mathrm{supp}(\mu)) = n+2$ and p_{n+2} is necessarily orthogonal with respect to μ. Define w_j, $j = 1, \ldots, n+2$ exactly as in Theorem 2.1 with n replaced by $n+1$ throughout. Then it follows that p_{n+1} is orthogonal with respect to μ and by Theorem 2.4, the weights $w_j > 0$, $j = 1, \ldots, n+2$ are uniquely determined because two points, x_1 and x_2, in $\mathrm{supp}(\mu)$ are known. The orthogonality of p_n with respect to μ follows easily from (4.8) since

$$\int x^\ell p_{n+2}(x)\,\mathrm{d}\mu(x) - \int x^\ell (x + a_n) p_{n+1}(x)\,\mathrm{d}\mu(x) = b_n \int x^\ell p_n(x)\,\mathrm{d}\mu(x),$$

and both integrals on the left-hand side of this equation are zero for $\ell = 0, 1, \ldots, n-1$. This completes the proof. $\square$

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